

# Stability of ferromagnetism in the Hubbard model on the kagomé lattice

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The Hubbard model on the kagomé lattice has highly degenerate ground states (the flat lowest band) in the corresponding single-electron problem and exhibits the so-called flat-band ferromagnetism in the many-electron ground states as was found by Mielke. Here we study the model obtained by adding extra hopping terms to the above model. The lowest single-electron band becomes dispersive, and there is no band gap between the lowest band and the other band. We prove that, at half-filling of the lowest band, the ground states of this perturbed model remain saturated ferromagnetic if the lowest band is nearly flat.

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Numerous studies have been made on the Hubbard model, a tight-binding model of electrons with on-site interactions, to understand mechanisms for ferromagnetism in itinerant electron systems in a simplified situation [1, 2, 3, 4]. Recently, Mielke [5] and Tasaki [6] brought a significant breakthrough in the field by proving that certain classes of Hubbard models have ferromagnetic ground states. These models in common have multi single-electron bands containing dispersionless bands, and are called flat-band Hubbard models.

Although these flat-band Hubbard models shed light on the role of the Coulomb interaction in generating ferromagnetism, the models with completely flat bands are still singular. It is desirable to clarify whether the flat-band ferromagnetism is stable against perturbations which turn the flat bands into dispersive bands. As for Tasaki's version of flat-band Hubbard models, local stability of the ferromagnetic ground state in perturbed nearly-flat-band models was proved [7]. Tasaki also gave a concrete example of nearly-flat-band Hubbard models in which he could prove that the ground states are ferromagnetic [8, 9]. See [10, 11] for related results.

As for Mielke's version of flat-band Hubbard models, on the other hand, there have been no rigorous results about stability (or instability) of ferromagnetism in perturbed nearly-flat-band models [12]. Here we note that there are essential differences between Mielke's and Tasaki's models. Mielke's models have simple structures where all the lattice sites are identical, while Tasaki's models have two different kinds of lattice sites. Reflecting the lattice structures, there are no band gaps in Mielke's models while there are finite band gaps in Tasaki's models. We stress that the problem of stability of ferromagnetism is much more subtle and difficult in Mielke's models, where one might encounter various low energy excitation modes which arise from the gapless nature of the band structures.

In this Letter, we treat the model obtained by adding hopping terms to the Hubbard model on the kagomé lattice, a typical example of Mielke's models [13]. The added perturbation destroys flatness of the band, but the band structure remains gapless. We prove that our model has saturated ferromagnetic ground states at half-

filling of the lowest band, provided that the lowest band is nearly flat.

*Definition.* We first define the reference triangular lattice  $\mathcal{L}$  as

$$\mathcal{L} = \left\{ n_1 \nu_1 + n_2 \nu_2 \mid \begin{array}{l} n_i \in \mathbb{Z} \text{ and } 0 \leq n_i < L \\ \text{for } i = 1, 2 \end{array} \right\}, \quad (1)$$

where  $\nu_1 = (1, 0)$ ,  $\nu_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ , and  $L$  is a positive integer. For each  $\alpha \in \mathcal{L}$  we define

$$C_\alpha^1 = \{x = n_1 \frac{\nu_1}{2} + n_2 \frac{\nu_2}{2} \mid n_1, n_2 \in \mathbb{Z}, |x - \alpha| = \frac{1}{2}\}, \quad (2)$$

$$C_\alpha^2 = \{x = n_1 \frac{\nu_1}{2} + n_2 \frac{\nu_2}{2} \mid n_1, n_2 \in \mathbb{Z}, |x - \alpha| = \frac{\sqrt{3}}{2}\}, \quad (3)$$

and  $C_\alpha = C_\alpha^1 \cup C_\alpha^2$  [14]. Then, the kagomé lattice  $\Lambda$  can be constructed as  $\Lambda = \cup_{\alpha \in \mathcal{L}} C_\alpha$ , where a site  $x \in \Lambda$  is generally counted four times in different  $C_\alpha$ . (See Fig.1.) Our lattice has open boundaries. Periodic lattices can be also treated with extra technical complication. We denote by  $\Phi_0$  the state with no electrons and denote by  $c_{x,\sigma}$  and  $c_{x,\sigma}^\dagger$  the annihilation and the creation operators, respectively, of an electron with spin  $\sigma$  at site  $x$  in  $\Lambda$ . These operators satisfy the usual fermion anticommutation relations. The number operator of an electron with spin  $\sigma$  at site  $x$  is defined as  $n_{x,\sigma} = c_{x,\sigma}^\dagger c_{x,\sigma}$ . The total spin operators  $\mathbf{S}_{\text{tot}} = (S_{\text{tot}}^{(1)}, S_{\text{tot}}^{(2)}, S_{\text{tot}}^{(3)})$  are defined as  $S_{\text{tot}}^{(i)} = \frac{1}{2} \sum_{x \in \Lambda} \sum_{\sigma, \tau = \uparrow, \downarrow} c_{x,\sigma}^\dagger p_{\sigma\tau}^{(i)} c_{x,\tau}$  for  $i = 1, 2, 3$ , where  $p^{(i)} = [p^{(i)}]_{\sigma, \tau = \uparrow, \downarrow}$  are the Pauli matrices. We denote by  $S_{\text{tot}}(S_{\text{tot}} + 1)$  the eigenvalue of  $(\mathbf{S}_{\text{tot}})^2$ .

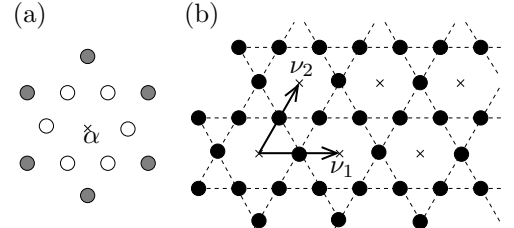


FIG. 1: (a) Local lattice  $C_\alpha = C_\alpha^1 \cup C_\alpha^2$ . The open and gray circles represent sites in  $C_\alpha^1$  and  $C_\alpha^2$ , respectively. (b) Lattice  $\Lambda$ .

Let us define our Hubbard Hamiltonian on  $\Lambda$ . First, for each  $x \in C_\alpha^2$  we define fermion operator  $b_{(\alpha,x),\sigma} = c_{x,\sigma} + \sum_{y \in C_\alpha^1; |y-x|=\frac{1}{2}} c_{y,\sigma}$ . We also define  $a_{\alpha,\sigma} = \sum_{y \in C_\alpha^1} \mu[\alpha, y] c_{y,\sigma}$  for  $\alpha \in \mathcal{L}$ , where coefficients  $\mu[\alpha, y]$  take either  $+1$  or  $-1$  and are chosen so that  $\mu[\alpha, y] \mu[\alpha, y'] = -1$  whenever  $|y - y'| = \frac{1}{2}$ . To each  $C_\alpha$  we associate the local Hamiltonian

$$\begin{aligned} \mathcal{H}_\alpha = & -s \sum_{\sigma=\uparrow, \downarrow} a_{\alpha,\sigma}^\dagger a_{\alpha,\sigma} + \frac{t}{3} \sum_{\sigma=\uparrow, \downarrow} \sum_{x \in C_\alpha^2} b_{(\alpha,x),\sigma}^\dagger b_{(\alpha,x),\sigma} \\ & + \frac{U}{4} \sum_{x \in C_\alpha} n_{x,\uparrow} n_{x,\downarrow}, \end{aligned} \quad (4)$$

where  $s$ ,  $t$  and  $U$  are positive parameters. Then, the Hubbard Hamiltonian on the whole lattice  $\Lambda$  is defined as  $\mathcal{H} = \sum_{\alpha \in \mathcal{L}} \mathcal{H}_\alpha$ .

*Remarks.* It is possible to rewrite  $\mathcal{H}$  into the standard form as  $\mathcal{H} = \sum_{\sigma} \sum_{x,y \in \Lambda} t_{xy} c_{x,\sigma}^\dagger c_{y,\sigma} + \sum_{x \in \Lambda} U_x n_{x,\uparrow} n_{x,\downarrow}$ , where the model parameters are given by  $U_x = U$ , and  $t_{xy} = 2(t-s)$  if  $x = y$ ,  $t_{xy} = t+s$  if  $|x-y| = 1/2$ ,  $t_{xy} = -s$  if  $|x-y| = \sqrt{3}/2$ ,  $t_{xy} = s$  if  $|x-y| = 1$  and  $x, y \in C_\alpha^1$  with some  $\alpha$ , and  $t_{xy} = 0$  otherwise, except for the sites close to the boundary. The single-electron dispersion relations (calculated in the model with periodic boundary conditions) are given by  $E_1(k) = -2s(3 - e(k))$ ,  $E_2(k) = t(3 - \sqrt{3+2e(k)})$ , and  $E_3(k) = t(3 + \sqrt{3+2e(k)})$  with  $e(k) = \cos k_1 + \cos k_2 + \cos(k_1 - k_2)$ , where  $k = k_1 \nu_1^* + k_2 \nu_2^*$  is the wave vector expanded in terms of reciprocal-lattice vectors  $\nu_1^* = (1, -\frac{1}{\sqrt{3}})$  and  $\nu_2^* = (0, \frac{2}{\sqrt{3}})$ . Note that  $E_1(0) = E_2(0) = 0$ , which means that there is no gap between the lowest and the second lowest bands.

One readily finds that  $\{a_{\alpha,\sigma}, b_{(\beta,x),\sigma}^\dagger\} = 0$  for any  $\alpha, \beta \in \mathcal{L}$  and  $x \in C_\beta^2$ . This implies that  $\{a_{\alpha,\sigma}^\dagger \Phi_0\}_{\alpha \in \mathcal{L}}$  spans the space corresponding to the lowest band.

If we set  $s = 0$ , our model has highly degenerate single-electron ground states, and becomes essentially the flat-band model of Mielke's (although there is a difference in boundary conditions). In this case, the model exhibits flat-band ferromagnetism for all positive values of  $U$ . In the model with  $s > 0$ , the situation is quite different because double occupancies of lower energy states, which destroy the ferromagnetic order, may reduce the total energy of the system. It is indeed easy to prove that the ground states of our model has  $S_{\text{tot}} = 0$  (or  $\frac{1}{2}$ ) for  $U = 0$ , and cannot exhibit saturated ferromagnetism for sufficiently small  $U$ . (See, for example, Sect. 3.3 of [4]). The following theorem establishes that the ferromagnetic ground states are stable for sufficiently large  $t$  and  $U$  when the electron number is  $|\mathcal{L}|$ .

*Main theorem.* Consider the Hubbard model defined as above with the electron number  $|\mathcal{L}|$ . Then, there exist critical values  $(t/s)_c$  and  $(U/s)_c$ , independent of the lattice size, such that, if both  $t/s > (t/s)_c$  and  $U/s > (U/s)_c$  are satisfied, the ground states of the model have  $S_{\text{tot}} = |\mathcal{L}|/2$ . Furthermore the ground state is unique up to the degeneracy due to the rotational symmetry.

In Tasaki's models, the stability of ferromagnetism may be, at least at a heuristic level, understood as a consequence of the band gap separating the lowest nearly flat band from other bands. The band gap enforces the electrons to occupy the lowest band while the interaction rules out double occupancies of sites. Then the situation is almost as in the flat-band models, and the systems exhibit ferromagnetism. To Mielke's models, which have no band gaps, the above argument does not apply, and the origin of the stability of ferromagnetism seems more subtle. Nevertheless, our proof is based on essentially the same philosophy as that of Tasaki's proof in [8]. Namely, we first establish ferromagnetism in a local model described by  $\mathcal{H}_\alpha$ , and then show that these local ferromagnetism can be "connected", which results in macroscopic ferromagnetism in the whole system. The results of the analysis of  $\mathcal{H}_\alpha$  are summarized in the following lemma.

*Lemma.* If  $t/s$  and  $U/s$  are sufficiently large, the minimum eigenvalue of  $\mathcal{H}_\alpha$  is  $-6s$  and any eigenstate  $\Phi$  belonging to this eigenvalue is written as

$$\Phi = a_{\alpha,\uparrow}^\dagger \Phi_\uparrow + a_{\alpha,\downarrow}^\dagger \Phi_\downarrow, \quad (5)$$

with appropriate states  $\Phi_\uparrow$  and  $\Phi_\downarrow$ . Furthermore it satisfies

$$c_{x,\downarrow} c_{x,\uparrow} \Phi = 0 \quad (6)$$

for all  $x \in C_\alpha$ .

*Proof of Lemma.* Since all the local Hamiltonians are the translated copies of  $\mathcal{H}_0$ , it suffices to prove the lemma for  $\alpha = 0$ . From now on, for convenience, we identify  $C_0^1$  and  $C_0^2$  with  $\{0, 2, \dots, 10\}$  and  $\{1, 3, \dots, 11\}$ , respectively (we first label  $(\frac{1}{2}, 0)$  as 0, then label the rest sites as  $1, \dots, 11$  in the clockwise order).

We start by solving a single-electron problem for  $\mathcal{H}_0$ . Let  $I = \{0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pi\}$ . Then the eigenvalues are given by

$$\varepsilon_1(p) = \begin{cases} 0 & \text{if } p \in I \setminus \{\pi\}; \\ -6s & \text{if } p = \pi, \end{cases} \quad (7)$$

and  $\varepsilon_2(p) = \frac{t}{3}(3 + 2\cos p)$  with  $p \in I$ . The eigenstate corresponding to  $\varepsilon_1(p)$  is expressed as  $d_{p,\sigma}^\dagger \Phi_0$  with

$$d_{p,\sigma} = \frac{1}{\sqrt{6(3+2\cos p)}} \sum_{l=0}^5 e^{ipl} (c_{2l,\sigma} - c_{2l-1,\sigma} - c_{2l+1,\sigma}) \quad (8)$$

(where  $c_{-1,\sigma}$  is regarded as  $c_{11,\sigma}$ ). Note that the set  $\{d_{p,\sigma}^\dagger \Phi_0\}_{p \in I}$  is orthonormal since  $\{d_{p,\sigma}, d_{p',\sigma}^\dagger\} = \delta_{p,p'}$ .

We consider many-electron problem for  $\mathcal{H}_0$ , first in the limit  $t, U \rightarrow \infty$ . Let  $\Phi$  be a state on  $C_0$  which has a finite energy in this limit. Since all  $\varepsilon_2(p)$  are infinite in the limit  $t \rightarrow \infty$ ,  $\Phi$  must be expanded as

$$\Phi = \sum_{I_\uparrow, I_\downarrow \subset I} g(I_\uparrow; I_\downarrow) \Phi(I_\uparrow; I_\downarrow) \quad (9)$$

with complex coefficients  $g(I_\uparrow; I_\downarrow)$ , where

$$\Phi(I_\uparrow; I_\downarrow) = \prod_{p \in I_\uparrow} d_{p,\uparrow}^\dagger \prod_{p' \in I_\downarrow} d_{p',\downarrow}^\dagger \Phi_0. \quad (10)$$

Here, and throughout the present Letter, the products are ordered in such a way that  $d_{p,\uparrow}^\dagger$  (*resp.*  $d_{p',\downarrow}^\dagger$ ) is always on the left of  $d_{p',\uparrow}^\dagger$  (*resp.*  $d_{p,\downarrow}^\dagger$ ) if  $p < p'$ . Since the on-site interaction  $n_{x,\uparrow} n_{x,\downarrow} = c_{x,\uparrow}^\dagger c_{x,\downarrow}^\dagger c_{x,\downarrow} c_{x,\uparrow}$  is positive semidefinite, the state  $\Phi$  in the form of (9) must further satisfy

$$\sum_{I_\uparrow, I_\downarrow \subset I} g(I_\uparrow; I_\downarrow) c_{x,\downarrow} c_{x,\uparrow} \Phi(I_\uparrow; I_\downarrow) = 0 \quad (11)$$

for any  $x \in C_0$  in order to have finite energy in the limit  $U \rightarrow \infty$ . From (7), (9) and (10) one finds that the expectation value of  $\mathcal{H}_0$  for the state  $\Phi$  is  $E_\Phi = (\Phi, \mathcal{H}_0 \Phi) / (\Phi, \Phi) = -6s + 6sF \|\Phi\|^{-2}$ , with  $F = \sum_{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}} (|g(I_\uparrow; I_\downarrow)|^2 - |g(I_\uparrow^\pi; I_\downarrow^\pi)|^2)$  and  $\|\Phi\|^2 = \left( \sum_{I_\uparrow, I_\downarrow \subset I} |g(I_\uparrow; I_\downarrow)|^2 \right)$  [15], where coefficients  $g$  should satisfy the condition (11). Here, and in what follows, we abbreviate  $I_\sigma \cup \{p\}$  as  $I_\sigma^p$  for  $p \in I$ . In the following, we will show  $F \geq 0$ . This implies  $E_\Phi \geq -6s$  since  $s > 0$ .

To prove  $F \geq 0$ , we first derive conditions on  $g$  imposed by eq. (11). If we denote  $(\varphi_x^{(p)})^* = \{c_{x,\sigma}, d_{p,\sigma}^\dagger\}$ , the left-hand-side of eq. (11) becomes

$$\begin{aligned} & \sum_{I_\uparrow, I_\downarrow \subset I; |I_\uparrow| \geq 1, |I_\downarrow| \geq 1} g(I_\uparrow; I_\downarrow) \sum_{p \in I_\uparrow} \sum_{p' \in I_\downarrow} (-1)^{|I_\uparrow|-1} S_{I_\uparrow}^p S_{I_\downarrow}^{p'} \\ & \quad \times (\varphi_x^{(p)})^* (\varphi_x^{(p')})^* \Phi(I_\uparrow \setminus \{p\}; I_\downarrow \setminus \{p'\}) \\ & = \sum_{p, p' \in I} (\varphi_x^{(p)})^* (\varphi_x^{(p')})^* \\ & \quad \times \sum_{I_\uparrow \subset I \setminus \{p\}} \sum_{I_\downarrow \subset I \setminus \{p'\}} (-1)^{|I_\uparrow|} S_{I_\uparrow}^p S_{I_\downarrow}^{p'} g(I_\uparrow^p; I_\downarrow^{p'}) \Phi(I_\uparrow; I_\downarrow) \\ & = \sum_{I_\uparrow, I_\downarrow \subset I} \sum_{p, p' \in I} (\varphi_x^{(p)})^* (\varphi_x^{(p')})^* \tilde{g}(I_\uparrow^p; I_\downarrow^{p'}) \Phi(I_\uparrow; I_\downarrow), \quad (12) \end{aligned}$$

where  $S_{I_\sigma}^p$ , which corresponds to a sign factor coming from exchange of the fermion operators, equals 1 if  $\sum_{p' \in I_\sigma; p' < p} 1$  is even and  $-1$  otherwise. In the final expression of (12), we introduced subsidiary coefficients  $\tilde{g}$  defined as  $\tilde{g}(I_\uparrow^p; I_\downarrow^{p'}) = 0$  if  $p \in I_\uparrow$  or  $p' \in I_\downarrow$  and  $\tilde{g}(I_\uparrow^p; I_\downarrow^{p'}) = (-1)^{|I_\uparrow|} S_{I_\uparrow}^p S_{I_\downarrow}^{p'} g(I_\uparrow^p; I_\downarrow^{p'})$  otherwise. Therefore,  $c_{x,\downarrow} c_{x,\uparrow} \Phi = 0$  holds if and only if  $\sum_{p, p' \in I} (\varphi_x^{(p)})^* (\varphi_x^{(p')})^* \tilde{g}(I_\uparrow^p; I_\downarrow^{p'}) = 0$  for any  $I_\uparrow, I_\downarrow \subset I$ . Taking the sum of this equation over  $x \in C_0^1$ , we find that  $\sum_{p \in I} \frac{1}{(3+2\cos p)} \tilde{g}(I_\uparrow^p; I_\downarrow^{-p}) = 0$  and similarly taking the sum over  $x \in C_0^2$ , we find that  $\sum_{p \in I} \frac{(1+\cos p)}{(3+2\cos p)} \tilde{g}(I_\uparrow^p; I_\downarrow^{-p}) = 0$  (where we identified  $-\pi$  with  $\pi$ ). By eliminating  $\tilde{g}(I_\uparrow^0; I_\downarrow^0)$  from these two equa-

tions, we obtain

$$\begin{aligned} \tilde{g}(I_\uparrow^\pi; I_\downarrow^\pi) &= -\frac{1}{16} \tilde{g}(I_\uparrow^\pi; I_\downarrow^{-\frac{\pi}{3}}) - \frac{1}{16} \tilde{g}(I_\uparrow^{-\frac{\pi}{3}}; I_\downarrow^\pi) \\ &\quad - \frac{3}{8} \tilde{g}(I_\uparrow^{\frac{2\pi}{3}}; I_\downarrow^{-\frac{2\pi}{3}}) - \frac{3}{8} \tilde{g}(I_\uparrow^{-\frac{2\pi}{3}}; I_\downarrow^{\frac{2\pi}{3}}). \quad (13) \end{aligned}$$

Our analysis below relies heavily on this condition.

For a subset  $I_\downarrow$  of  $I$ , we define  $\bar{I}_\downarrow = \{-p \mid p \in I_\downarrow\}$  and denote by  $N(I_\uparrow; I_\downarrow)$  the number of elements in  $I_\uparrow \cap \bar{I}_\downarrow \cap (I \setminus \{0, \pi\})$ . Condition (13) relates  $\tilde{g}(I_\uparrow; I_\downarrow)$  with  $I_\uparrow, I_\downarrow$  such that  $N(I_\uparrow; I_\downarrow) = r$  and  $\tilde{g}(I_\uparrow'; I_\downarrow')$  with  $I_\uparrow', I_\downarrow'$  such that  $N(I_\uparrow'; I_\downarrow') = r+1$ . This motivates us to decompose  $F$  as  $F = F' + \sum_{r=0}^4 F_r$ , where

$$F_r = \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r+1}} |g(I_\uparrow; I_\downarrow)|^2 - \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r}} |g(I_\uparrow^\pi; I_\downarrow^\pi)|^2, \quad (14)$$

$$F' = \sum_{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; N(I_\uparrow; I_\downarrow) = 0} |g(I_\uparrow; I_\downarrow)|^2. \quad (15)$$

Since the term  $F'$  is apparently non-negative,  $F \geq 0$  is implied by  $F_r \geq 0$  for  $r = 0, \dots, 4$ .

We shall prove that  $F_r \geq 0$  by using (13). For a pair of  $I_\uparrow^\pi$  and  $I_\downarrow^\pi$  such that  $N(I_\uparrow^\pi; I_\downarrow^\pi) = r$ , the number of non-zero  $\tilde{g}$  in the right hand side of (13) is, by the definition, at most  $4 - r$ , and thus for such a pair we have [16]

$$|\tilde{g}(I_\uparrow^\pi; I_\downarrow^\pi)|^2 \leq \frac{9}{64} (4-r) \sum_{p \in I \setminus \{0, \pi\}} |\tilde{g}(I_\uparrow^p; I_\downarrow^{-p})|^2. \quad (16)$$

Then, we find that

$$\begin{aligned} & \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r}} |g(I_\uparrow^\pi; I_\downarrow^\pi)|^2 = \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r}} |\tilde{g}(I_\uparrow^\pi; I_\downarrow^\pi)|^2 \\ & \leq \frac{9}{64} (4-r) \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r}} \sum_{p \in I \setminus \{0, \pi\}} |\tilde{g}(I_\uparrow^p; I_\downarrow^{-p})|^2 \\ & = \frac{9}{64} (4-r)(r+1) \sum_{\substack{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; \\ N(I_\uparrow; I_\downarrow) = r+1}} |g(I_\uparrow; I_\downarrow)|^2 \\ & \leq \frac{27}{32} \sum_{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; N(I_\uparrow; I_\downarrow) = r+1} |g(I_\uparrow; I_\downarrow)|^2. \quad (17) \end{aligned}$$

To get the third line, we have used the fact that, for  $I_\uparrow$  and  $I_\downarrow$  such that  $N(I_\uparrow; I_\downarrow) = r+1$ , there are  $r+1$  elements  $p$  in  $I \setminus \{0, \pi\}$  for which we can find suitable subsets  $I_\uparrow'$  and  $I_\downarrow'$  such that  $\{p\} \cup I_\uparrow' = I_\uparrow$  and  $\{-p\} \cup I_\downarrow' = I_\downarrow$ . To obtain the final inequality, we have used  $(4-r)(r+1) \leq 6$  for  $0 \leq r \leq 4$ . By using (17) we obtain

$$F_r \geq \frac{5}{32} \sum_{I_\uparrow, I_\downarrow \subset I \setminus \{\pi\}; N(I_\uparrow; I_\downarrow) = r+1} |g(I_\uparrow; I_\downarrow)|^2 \geq 0. \quad (18)$$

We therefore conclude that  $F \geq 0$ . The above analysis also shows that the equality  $F = 0$  holds only when  $F'$  and  $F_r$  are vanishing, i.e.,  $g(I_\uparrow; I_\downarrow) = 0$  for any pair of  $I_\uparrow$  and  $I_\downarrow$  such that  $\pi \in I_\uparrow \cap I_\downarrow$  or  $\pi \notin I_\uparrow \cup I_\downarrow$ .

In other words we have shown that  $E_\Phi \geq -6s$  for any  $\Phi$  and that any  $\Phi$  attaining the minimum expectation value  $-6s$  is written as

$$\Phi = \sum_{I_\uparrow, I_\downarrow \subset I; \pi \in I_\uparrow \cup I_\downarrow, \pi \notin I_\uparrow \cap I_\downarrow} g(I_\uparrow; I_\downarrow) \Phi(I_\uparrow; I_\downarrow) \quad (19)$$

and further satisfies the finite energy condition (11). One finds that such minimizing  $\Phi$  indeed exists by testing  $d_{\pi, \uparrow}^\dagger \Phi_0$  or  $\prod_{p \in I} d_{p, \uparrow}^\dagger \Phi_0$ . By construction such  $\Phi$  is an eigenstate of  $\mathcal{H}_0$  as well as  $-s \sum_\sigma a_{0, \sigma}^\dagger a_{0, \sigma}$ . Since it is known to be the lowest energy state of  $\mathcal{H}_0$  in the limit  $t, U \rightarrow \infty$ , the continuity of energy implies that such  $\Phi$  is the lowest energy state of  $\mathcal{H}_0$  for sufficiently large  $t/s$  and  $U/s$ . It is also easy to check that such  $\Phi$  has the properties stated in Lemma. (Note that  $d_{\pi, \sigma} = \frac{\mu[0, 0]}{\sqrt{6}} a_{0, \sigma}$ .) This completes the proof of Lemma. ■

*Proof of Theorem.* We assume that the values of  $t/s$  and  $U/s$  are large enough for the statement in Lemma to hold. We note that how large  $t/s$  and  $U/s$  should be is independent of the size of  $\Lambda$ , because Lemma is concerned with the local Hamiltonian.

From Lemma we find that the eigenvalue of  $\mathcal{H}$  is bounded below by  $-6s|\mathcal{L}|$ , while, by taking  $\Phi_f = \prod_{\alpha \in \mathcal{L}} a_{\alpha, \uparrow}^\dagger \Phi_0$  as a variational state, we find that  $-6s|\mathcal{L}|$  is an upper bound on the ground state energy. Therefore, the ground state energy is  $-6s|\mathcal{L}|$ , and  $\Phi_f$  and its  $SU(2)$  rotations are among the corresponding eigenstates. It is apparent that these states have  $S_{\text{tot}} = |\mathcal{L}|/2$ .

The remaining task is to prove the uniqueness. Let  $\Phi_G$  be an arbitrary ground state of  $\mathcal{H}$ . Lemma implies that the ground state energy is attained if and only if  $\mathcal{H}_\alpha \Phi_G = -6s\Phi_G$  for all  $\alpha \in \mathcal{L}$ . Thus  $\Phi_G$  must satisfy the conditions stated in Lemma.

The condition (5) implies that  $\Phi_G$  is expressed as [17]

$$\Phi_G = \sum_{\{\sigma\}} \varphi(\{\sigma\}) \prod_{\alpha \in \mathcal{L}} a_{\alpha, \sigma_\alpha}^\dagger \Phi_0, \quad (20)$$

where  $\{\sigma\}$  is a shorthand for a spin configuration  $\{\sigma_\alpha\}_{\alpha \in \mathcal{L}}$ , the summation is over  $\sigma_\alpha = \uparrow, \downarrow$  for all  $\alpha \in \mathcal{L}$ , and  $\varphi(\{\sigma\})$  is a complex coefficient.

Let us impose the condition (6) on  $\Phi_G$  in the form of (20). Let  $\beta$  and  $\gamma$  be nearest neighbour points in  $\mathcal{L}$ , i.e.,  $|\beta - \gamma| = 1$ , and let  $m(\beta, \gamma) \in \Lambda$  be the site located at the mid-point between  $\beta$  and  $\gamma$ . It is easy to see that  $\{c_{m(\beta, \gamma), \sigma} a_{\alpha, \sigma}^\dagger\}$  is nonvanishing if  $\alpha = \beta$  or  $\gamma$ , and is vanishing otherwise. Then, it follows from the condition  $c_{m(\beta, \gamma), \downarrow} c_{m(\beta, \gamma), \uparrow} \Phi_G = 0$  that  $\varphi(\{\sigma\}) = \varphi(\{\tau\})$  for any pair of spin configurations  $\{\sigma\}$  and  $\{\tau\}$  satisfying that  $\sigma_\beta = \tau_\gamma$ ,  $\sigma_\gamma = \tau_\beta$ , and  $\sigma_\alpha = \tau_\alpha$  for  $\alpha \neq \beta, \gamma$ . Examining  $c_{m(\beta, \gamma), \downarrow} c_{m(\beta, \gamma), \uparrow} \Phi_G = 0$  for all the pairs of nearest neighbour points in  $\mathcal{L}$ , we find that  $\varphi(\{\sigma\}) = \varphi(\{\tau\})$  whenever  $\sum_{\alpha \in \mathcal{L}} \sigma_\alpha = \sum_{\alpha \in \mathcal{L}} \tau_\alpha$ . Therefore  $\Phi_G$  is written as  $\Phi_G = \sum_{M=0}^{|\mathcal{L}|} \varphi_M (S_{\text{tot}}^-)^M \Phi_f$ , where  $\varphi_M$  are new coefficients and the spin lowering operator  $S_{\text{tot}}^-$  is defined by  $S_{\text{tot}}^- = \sum_{x \in \Lambda} c_{x, \downarrow}^\dagger c_{x, \uparrow}$ . This completes the proof. ■

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  - [13] It is possible to prove stability of ferromagnetism in Mielke's models on other lattices such as a regular lat-

tice of corner-sharing tetrahedra. The details will appear elsewhere.

- [14] Here  $|\cdot|$  denotes the Euclidean norm. We use the same symbol  $|X|$  to denote the number of elements in a set  $X$ .
- [15] Note that  $\Phi(I_\uparrow; I_\downarrow)$  are the eigenstates of  $-s \sum_\sigma a_{0, \sigma}^\dagger a_{0, \sigma}$  and the set of these states is orthonormal.
- [16] We have used inequality  $\left| \sum_{i=1}^N z_i \right|^2 \leq N \sum_{i=1}^N |z_i|^2$  which follows from the Schwarz inequality.
- [17] Proof: Since all  $a_{\alpha, \sigma}^\dagger \Phi_0$  with  $\alpha \in \mathcal{L}$  are linearly independent, we can form a basis of the single-electron Hilbert space on  $\Lambda$  by adding  $|\Lambda| - |\mathcal{L}|$  linearly independent states. We denote these states by  $f_{i, \sigma}^\dagger \Phi_0$  with  $i \in \mathcal{I}$  where  $\mathcal{I}$  is some index set with  $|\mathcal{I}| = |\Lambda| - |\mathcal{L}|$ . Then, the set of states  $\left( \prod_{\alpha \in \mathcal{L}_\uparrow} a_{\alpha, \uparrow}^\dagger \right) \left( \prod_{\beta \in \mathcal{L}_\downarrow} a_{\beta, \downarrow}^\dagger \right) \left( \prod_{i \in \mathcal{I}_\uparrow} f_{i, \uparrow}^\dagger \right) \left( \prod_{j \in \mathcal{I}_\downarrow} f_{j, \downarrow}^\dagger \right) \Phi_0$  with  $\mathcal{L}_\uparrow, \mathcal{L}_\downarrow \subset \mathcal{L}$  and  $\mathcal{I}_\uparrow, \mathcal{I}_\downarrow \subset \mathcal{I}$  satisfying  $\sum_\sigma (|\mathcal{L}_\sigma| + |\mathcal{I}_\sigma|) = N_e$  is a basis of  $N_e$ -electron Hilbert space. We suppose that  $\Phi_G$  is expanded with respect to this basis and then use condition (5).